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An Example of the Effect of Irreversible Microscopic Dynamics on the Macroscopic Behavior of a Large System

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The original Kac ring model is examined for the situation in which some of the scatterers are non-time-reversal invariant. It is shown that the system tends to absolute equilibrium (no fluctuations) even in the limit of very small density of these anomalous scatterers.

KEY WORDS: Absolute equilibrium; anomalous scatterer; Kac ring model; microscopic irreversibility; time-reversal- and non-time-reversal-invariant.

1. INTRODUCTION

The one-dimensional ring model proposed by Kac[(1)] and developed by others[(2)] has been very successful in explicating the irreversible behavior of a large system whose underlying (microscopic) dynamics is time-reversal-invariant. Various phenomena, assumed or believed true for real (three-dimensional) systems, can be demonstrated explicitly for the Kac ring model. These phenomena include the approach to equilibrium in terms of the reduction of the Liouville equation to the Boltzmann equation via the Pauli master equation,[(2)] the interdependence of small systems (clocks),[(3)] and the behavior of fluctuations at equilibrium.[(4)] Until the present the model has been solved with a variety of microscopic dynamics, including both quantum mechanical and classical mechanical, with the proviso that the interaction be time-reversal invariant.[(5)] We propose here to examine the effect on the macroscopic behavior of the inclusion of a non-time-reversal-invariant part to the interaction. In particular, we shall investigate the behavior of the system in the limit of a very weak, non-time-reversal-invariant interaction. Such a limit will be seen to be akin to the limit of weak interaction in deriving

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the ideal gas equation (classical or quantum mechanical) using equilibrium statistical mechanics.[6]

The motivation for this investigation is twofold. First, there is the obvious connection to the real world as manifested by the $K\bar{K}$ non-time-reversal-invariant coupling to the baryons.[7] The object is to see what the introduction of even a vanishingly small density of such particles will do to the macroscopic behavior of the universe. The second reason is more general. We would like to investigate the behavior of an exactly soluble model (albeit, nonphysical) for as wide a range of interactions as possible. To this end, we turn our attention to a description of the proposed model in the next section.

2. THE TIME-REVERSIBLE MODEL

The original (classical) model assumed n particles (balls) with two internal states (black and white) and m randomly located scatterers which change the state of a particle when it passes them.[1] The boundary conditions are periodic; that is, the n particles are arranged in the form of a ring. More precisely, the state η of a particle is given by

$$n_p(t) = +1 \qquad \text{if the particle at position } p \text{ at time } t \text{ is white}$$

$$n_p(t) = -1 \qquad \text{if the particle at position } p \text{ at time } t \text{ is black}$$

The indicator for the position is written as

$$\epsilon_p = +1 \qquad \text{if there is no scatterer at position } p$$

$$\epsilon_p = -1 \qquad \text{if there is a scatterer at position } p$$

The equation of motion is

$$n_p(t+1) = \epsilon_{p+1}\eta_{p+1}(t) \tag{1}$$

or

$$n_p(t) = \epsilon_{p+1} \cdots \epsilon_{p+t}\eta_{p+t}(0) \tag{2}$$

indicating that a particle changes state when passing a scatterer. The macrostate of the system is described by the order parameter

$$\Gamma(t) = (1/n) \sum_{p=1}^{n} n_p(t) \tag{3}$$

The behavior of Γ as a function of time for the initial condition $\Gamma(0) = 1$ (all white balls) has been given elsewhere and will not be repeated here.[3,4] Rather, we shall use a geometric description of the temporal behavior of the system as presented in Ref. 4. This will enable us to include what we shall refer to as anomalous scatterers. Figure 1 of Ref. 4 illustrates the description for one model with $n = 78$ and $m = 8$ starting with all white balls at $t = 0$. (The X's denote white balls and the dots denote black balls.) It can be seen that the essential feature of the temporal behavior is given by the alternation of black and white parallelograms. It is thus not necessary to think of the model as having a discrete set of positions. We can therefore imagine Fig. 1 redrawn so that there is a continuous distribution on the p and t axes while preserving a fixed number of scatterers. These scatterers then serve to indicate where the boundaries of the parallelograms are. (The precise meaning of the continuous description in terms of the original model is not needed here and will be elaborated on in a future paper dealing with the relationship between the Kac ring model and the n-dimensional Ising model.) It is now possible to add anomalous or non-time-reversal-invariant scatterers to the set of ordinary or time-reversal-invariant scatterers and obtain a simple description of the temporal behavior of the new system. This perhaps unexpected behavior is given in the next section.

3. THE TIME-IRREVERSIBLE MODEL

Before constructing a general theorem concerning the behavior of Kac ring models with anomalous scatterers, we present a few case histories from which the general behavior will be almost self-evident and will require only simple arguments. We first define an anomalous scatterer as one which changes white balls to black balls but leaves black balls unchanged. The number of such scatterers will be denoted by m_1. It might be supposed that when n and m are both large with m/n fixed and m_1 small, the original model behavior would be recovered. We shall see, however, that such is not the case.

The following is immediately evident as a common feature of all five systems. Unlike the case of no anomalous scatterers in which there are fluctuations in equilibrium, it is seen that the presence of anomalous scatterers forces the system to reach an equilibrium configuration in which there is no fluctuation in T. We shall refer to this as absolute equilibrium. Furthermore, the Poincaré cycle is completely destroyed by the presence of even one anomalous scatterer. The above behavior may be thought of as a theorem, which we now prove by means of the following lemmas, given an initial condition of all white balls.

Definition. An interval is the distance on the p axis between any two adjacent scatterers.

Lemma 1. Parallelograms bounded by diagonal lines and vertical lines through adjacent scatterers alternate color if the scatterers are ordinary.

Parallelograms bounded on the left by a vertical line through an anomalous scatterer are black.

Proof. This follows directly from the time development picture, together with the definition of ordinary and anomalous scatterers.

Corollary. The interval directly to the right of an anomalous scatterer is black after a time equal to the length of the interval directly to the left of this scatterer.

Lemma 2. The time required for a column to stabilize its color (reach equilibrium) is the distance of the right side of the column from the nearest anomalous scatterer to the left.

Proof. The time required for a column directly to the right of an anomalous scatterer to stabilize its color (black) is equal to the width of the column. Since the parallelograms alternate in color to the right of this column until the next anomalous scatterer is reached, the time required for any column to stabilize its color is as stated.

Corollary. The maximum time for a system to come to equilibrium is $n - (m_1 - 1)$ for the case when all the anomalous scatterers are consecutive. Thus the system will always come to absolute equilibrium in a time less than or equal to n.

Theorem. An arbitrarily large but finite ring model with an arbitrarily large number of ordinary scatterers and an arbitrarily small number of anomalous scatterers will reach absolute equilibrium (constant value of Γ) in a time less than the largest distance between anomalous scatterers.

The theorem implies that there is a qualitative difference between the behavior of large ring models with m/n fixed and containing some or no anomalous scatterers. This is true for the Poincaré cycle, which is a manifestation of the reversibility of the underlying dynamics. The absence of a Poincaré cycle for a system containing anomalous scatterers merely reflects the microscopic irreversibility. However, the approach to equilibrium which is a consequence of the statistical nature of a large system is unaffected by a small number of anomalous scatterers. This is so because the relaxation time is given by $\tau = n/2m$, which is much smaller than the largest distance between anomalous scatterers ($>n/2m_1$). Thus, the system settles down to ordinary (fluctuating) equilibrium long before absolute equilibrium sets in.

4. CONCLUSION

The behavior of the foregoing model, while of interest in itself, has some implication for the behavior of a universe containing microscopic reactions which are not time-reversal invariant. The obvious application is to the case of the K–$\overline{K}$ coupling to the baryons. Of course, any conclusions about a three-dimensional system, especially one in which the interactions are far more complicated than those of the Kac ring model, are immediately suspect. Nevertheless, it is interesting to speculate on the nature of such conclusions.

The main effect of the inclusion of an arbitrarily small number of anomalous scatterers in the original Kac ring model is the destruction of the Poincaré cycle. If this behavior is symptomatic of all systems, then the obvious conclusion is that the presence of an arbitrarily small number of K–$\overline{K}$ particles, or alternatively, an arbitrarily small, time-irreversible branching ratio for K decay, will destroy the Poincaré cycle of the universe considered as a closed system. This would probably rule out oscillating universes as cosmological models.[8]

Another consequence of the destruction of the Poincaré cycle is a difference in the form of Γ in the limit of arbitrarily small number of anomalous scatterers and the form of Γ when no anomalous scatterers are present. In the first instance, a certain effect is present ($\Gamma \approx 0 = \text{const}$ for $t > n$) which is not present in the second instance $\Gamma(t = n) = \Gamma(t = 0)$. This is analogous to the statistical mechanical treatment of the ideal gas as the limit of a gas with arbitrarily weak interparticle interactions in contrast with a system for which the interparticle interactions are strictly zero. In the first instance, a certain effect is again present (Maxwell–Boltzmann distribution of velocities for $t >$ relaxation time) which is not present in the second instance (distribution of velocities equals initial distribution for all time).

REFERENCES

1. M. Kac, *Bull. Roy. Soc. Belgium* **42**:356 (1956).
2. M. Dresden, in *Studies of Statistical Mechanics*, de Boer and Uhlenbeck, eds. (North-Holland, Amsterdam, 1962), Vol. I, p. 303.
3. M. Coopersmith and G. Mandeville, *J. Stat. Phys.* **10**:391 (1974).
4. G. Mandeville and M. Coopersmith, *J. Stat. Phys.* **10**:406 (1974).
5. M. Dresden and F. Feiock, *J. Stat. Phys.* **4**:111 (1972).
6. F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965), p. 64.
7. T. D. Lee and C. S. Wu, *Ann. Rev. Nucl. Sci.* **16**:471 (1966).
8. W. Rindler, *Essential Relativity* (Van Nostrand Reinhold, New York, 1969), p. 261.